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SHARP MAXIMAL FUNCTION ESTIMATES AND H^p CONTINUITIES OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. It is studied that pointwise estimates and continuities on Hardy spaces of the pseudo-differential operators (PDOs for short) with the symbol in general Hörmander's classes. We get weighted weak-type $(1,1)$ estimate, weighted normal inequalities, (H^p, H^p) continuities and (H^p, L^p) continuities for the PDOs, where $0 < p \leq 1$.

Keywords: pseudo-differential operators, Hardy spaces, pointwise estimates, general Hörmander's classes, inequalities.

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1. Introduction

Let $m \in \mathbb{R}$, $0 \leq \varrho, \delta \leq 1$. A symbol $a(x,\xi)$ is said to be in the Hörmander $S^m_{\varrho,\delta}$ class as given in [21], if $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\varrho|\alpha|+\delta|\beta|},$$

for any multi-indices α, β. The pseudo-differential operators with symbol $a(x,\xi)$ is defined by the formula

$$T_a u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi) \hat{u}(\xi) d\xi, \tag{1}$$

where $\hat{u}$ denotes the fourier transform of u. An important topic on the pseudo-differential operators is to study the properties of these operators acting on some function spaces and some pointwise estimates for them. L^p regularity is a fundamental one which can be gotten by the complex interpolation between L^2-continuity and (L^∞, BMO)-continuity, see [12, 34, 35]. As we know, L^2-continuity of the pseudo-differential operators is sharp in terms of its order $m \leq -\frac{n}{2}\max\{\delta - \varrho, 0\}$, where $0 \leq \varrho \leq 1$ and $0 \leq \delta < 1$, see [20, 19]. However, it is not clear if the (L^∞, BMO)-continuity is sharp when $0 \leq \varrho < \delta < 1$, see [23, 27]. On the one hand, if $a(x,\xi) \in L^\infty S^m_\varrho$ with $m < -\frac{n}{2}(1-\varrho)$, the pseudo-differential operators are bounded on $L^\infty(\mathbb{R}^n)$ [23], which implies the (L^∞, BMO)-continuity. Here, $L^\infty S^m_\varrho$ denotes the rough Hörmander class whose constituent $a(x,\xi)$ obeys

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m-\varrho|\alpha|}.$$

Clearly, the relation $S^m_{\varrho,\delta} \subset L^\infty S^m_\varrho$ holds for any $m \in \mathbb{R}$, $1 \leq \varrho, \delta \leq 1$. On the other hand, there is a symbol a $a \in S^m_{\varrho,0}$ such that T_a dose not map L^∞ to BMO if $m > -\frac{n}{2}(1-\varrho)$, see [27].

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Recently, taking full advantage of the smooth of variate x, the author of the paper [40] prove that if $0 \leq \varrho \leq 1$, $0 \leq \delta < 1$ and $a(x,\xi) \in S^{-\frac{n}{2}(1-\varrho)}_{\varrho,\delta}$, the (L^∞, BMO)-continuity of the pseudo-differential operators T_a is true, and clearly it is sharp. Moreover, the L^p boundedness is studied as well.

Theorem 1.1 (See [40])**.** *Let $1 < p < \infty$, $0 \leq \varrho \leq 1$, $0 \leq \delta < 1$ and $a(x,\xi) \in S^m_{\varrho,\delta}$. If*

$$m \leq -n(1-\varrho)|\frac{1}{2} - \frac{1}{p}| - n\frac{\max\{\delta-\varrho, 0\}}{\max\{p, 2\}},$$

then

$$\|T_a u\|_{L^p} \lesssim \|u\|_{L^p}.$$

Clearly, the range of m in [1, Theorem 3.4] is revised when $2 < p < \infty$ and $0 \leq \varrho < \delta < 1$. For the case $1 < p < \infty$ and $0 \leq \delta \leq \varrho < 1$, we refer to [20, 35, 38].

It is a pity that the main idea is inapplicable to its dual operators T^*_a which is defined by the formula

$$T^*_a u(x) \quad = \quad \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(y,\xi) d\xi u(y) dy. \tag{2}$$

So, the (L^∞, BMO)-continuity of T^*_a has been understood so far [1] only if $a(y,\xi) \in S^{-\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$. However, one can get (H^1, L^1)-continuity of T^*_a under the condition $a(y,\xi) \in S^{-\frac{n}{2}(1-\varrho)}_{\varrho,\delta}$ (see Theorem 1.16). By complex interpolation, we have the following result.

Theorem 1.2. *Let $1 < p < \infty$, $0 \leq \varrho \leq 1$, $0 \leq \delta < 1$ and $a(x,\xi) \in S^m_{\varrho,\delta}$. If*

$$m \leq -n(1-\varrho)|\frac{1}{2} - \frac{1}{p}| - n\max\{\delta-\varrho, 0\}(1 - \frac{1}{\min\{p, 2\}}),$$

then

$$\|T^*_a u\|_{L^p} \lesssim \|u\|_{L^p}.$$

In this paper, the properties of pseudo-differential operator acting on Hardy spaces $H^p(\mathbb{R}^n)$ that is a right replacement for $L^p(\mathbb{R}^n)$ when $0 < p \leq 1$, and some pointwise estimates for these operators are investigated. Clearly, the $L^p (p \neq 2)$ continuity between T_a and T^*_a is different in terms of the order m. Based on this observation, both T_a and T^*_a will be considered in this paper.

For the sake of narration, it is necessary to introduce some notations firstly. For a function $u \in L^1_{loc}(\mathbb{R}^n)$, we define the Fefferman-Stein sharp maximal function and Hardy-Littlewood maximal function by:

$$M^\sharp u(x) = \sup_{x\in Q}\inf_c \frac{1}{|Q|}\int_Q |u(y)-c| dy \quad \text{and} \quad Mu(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |u(y)| dy$$

respectively, where c moves over all complex number, and Q containing x moves over all cubes with its sides parallel to the coordinate axes. For $\epsilon > 0$, let us denote $M^\sharp_\epsilon u = \big(M^\sharp(|u^\epsilon|)\big)^{1/\epsilon}$ and $M_\epsilon u = \big(M(|u^\epsilon|)\big)^{1/\epsilon}$.

The pointwise estimate of pseudo-differential operators in terms of $M^\sharp$ and M are given by many authors in [8, 22, 26, 27, 29, 39, 40]. We refer to [7, 4, 5] for the pointwise sparse bounds of these operators. Here, one is apt to state a result given in [28].

Theorem 1.3 (see [28])**.** *Let $1 < p \leq 2$, $0 < \varrho \leq \frac{p}{2}$ and $\varrho < 1$. If $a \in S^{-n(1-\varrho)/p}_{\varrho,\varrho}$, then*

$$M^\sharp(T_a f)(x) \lesssim M_p f(x).$$

Clearly, there is a restriction on the range of ϱ, δ and p, that is $0 < \varrho = \delta \le \frac{p}{2}$ with $\varrho < 1$ and $p \ne 1$. Recently, this restriction on ϱ, δ is extended to $0 \le \varrho = \delta < 1$ in [29, 32] and to $0 \le \varrho \le 1$, $0 \le \delta < 1$ when $p = 2$ in [40]. However, the case of $1 < p < 2$, $0 \le \varrho \le 1$, $0 \le \varrho < \delta < 1$ and $p = 1$, $0 \le \varrho \le 1$, $0 \le \delta < 1$ seems to be not clear. Particularly, there is no corresponding result in case $p = 1$, but a weaker version is obtained in [24].

Theorem 1.4 (See [24])**.** *Let $0 < \varrho \le 1$, $0 \le \delta < 1$ and $1 < p < \infty$. If $a \in S^{-n(1-\varrho)}_{\varrho,\delta}$, then*

$$M^\sharp(T_a f)(x) \lesssim M_p f(x).$$

The first main result of this paper is a generalization of Theorem 1.3. And the operator T_a^* is considered as well.

Theorem 1.5. *Let $0 \le \varrho \le 1$, $0 \le \delta < 1$ and $1 < p \le 2$. If $a \in S^{-n(1-\varrho)/p}_{\varrho,\delta}$, then*

$$M^\sharp(T_a f)(x) \lesssim M_p f(x).$$

If $a \in S^{-\frac{n}{p}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$, then

$$M^\sharp(T_a^* f)(x) \lesssim M_p f(x).$$

The second main result of this paper is extending p in Theorem 1.4 to the extreme case $p = 1$.

Theorem 1.6. *Let $0 < \varrho \le 1$, $0 \le \delta < 1$ and $0 < \epsilon < 1$. If $a \in S^{-n(1-\varrho)}_{\varrho,\delta}$, then*

$$M^\sharp_\epsilon(T_a f)(x) \lesssim Mf(x) \quad \text{and} \quad M^\sharp_\epsilon(T_a^* f)(x) \lesssim Mf(x).$$

Interesting that the order of T_a^* in Theorem 1.5 seem to be improved when $p = 1$. It is not clear that if the order of T_a^* can be improved in the case $1 < p \le 2$. Another interesting thing is that the second estimate holds with $a \in L^\infty S^{-n(1-\varrho)}_\varrho$ in case $0 < \varrho < 1$.

Theorem 1.7. *Let $0 < \varrho < 1$. If $a \in L^\infty S^{-n(1-\varrho)}_\varrho$ then*

$$M^\sharp(T_a^* f)(x) \lesssim Mf(x).$$

As we know, the pointwise estimates can give some weighted L^p inequalities. Recall that a nonnegative locally integrable function ω belongs to the class of Muckenhoupt A_p weights if there exists a constant $C > 0$ such that

$$\sup_{Q \subset \mathbb{R}^n} \big(\tfrac{1}{|Q|}\textstyle\int_Q \omega(x)dx\big)\big(\tfrac{1}{|Q|}\textstyle\int_Q \omega(x)^{\frac{1}{1-p}}dx\big)^{p-1} \le C, \quad \text{when} \quad 1 < p < \infty; \tag{3}$$

$$M\omega(x) \le C\omega(x) \quad \text{for almost all } x \in \mathbb{R}^n, \quad \text{when} \quad p = 1. \tag{4}$$

For $p = \infty$, one can define $A_\infty := \cup_{p>1} A_p$. The smallest constant appearing in (3) or (4) is called the A_p constant of ω which is denoted by $[\omega]_p$. The usual notation that

$$\|u\|^p_{L^p_\omega} = \int_{\mathbb{R}^n} |u(x)|^p \omega(x)dx \text{ and } \|u\|^p_{L^{p,\infty}_\omega} = \sup_{\lambda>0} \lambda^p \omega(x \in \mathbb{R}^n : |u(x)| > \lambda)$$

will be adopted in this paper. The weighted L^p estimates for pseudo-differential operators has been a topic extensively studied, specially in the 1980s [1, 8, 22, 28], later improved in [24, 25] in the late 2000s and revisited in [29, 40] recently.

Theorem 1.8 (See [40])**.** *Let $0 \le \varrho \le 1$, $0 \le \delta < 1$, $1 \le r \le 2$ and $a(x,\xi) \in S^{-\frac{n}{r}(1-\varrho)}_{\varrho,\delta}$. Suppose $\omega \in A_{p/r}$ with $r < p < \infty$. Then, there is a constant C independent of a and u, such that*

$$\|T_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega}. \tag{5}$$

Theorem 1.8 is proved by some interpolations between $r = 1$ and $r = 2$. In this paper, a new proof will be given.

By interpolation theory [6, Theorem 5.5.3] and the famous Fefferman-Stein's inequalities [11], we can write

$$\|M_\epsilon u\|_{L^p_\omega} \lesssim \|M^\sharp_\epsilon u\|_{L^p_\omega}, \quad \|M_\epsilon u\|_{L^{p,\infty}_\omega} \lesssim \|M^\sharp_\epsilon u\|_{L^{p,\infty}_\omega}$$

for $0 < \epsilon, p < \infty$ and $\omega \in A_\infty$, Theorem 1.5, Theorem 1.6 and Theorem 1.7 lead to the following weighted L^p inequalities.

Theorem 1.9. *Let $0 \le \varrho \le 1$, $0 \le \delta < 1$ and $1 \le r \le 2$. For any $r \le p < \infty$ ($1 < p < \infty$ if $r = 1$) and $\omega \in A_{p/r}$, if $a(x,\xi) \in S^{-\frac{n}{r}(1-\varrho)}_{\varrho,\delta}$, then*

$$\|T_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega}.$$

if $a \in S^{-\frac{n}{r}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$, then

$$\|T^*_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega}.$$

Theorem 1.10. *Let $0 < \varrho \le 1$, $0 \le \delta < 1$. For any $1 < p < \infty$ and $\omega \in A_p$, if $a \in S^{-n(1-\varrho)}_{\varrho,\delta}$, then*

$$\|T_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega} \text{ and } \|T^*_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega}.$$

For $p = 1$ and $\omega \in A_1$, if $a \in S^{-n(1-\varrho)}_{\varrho,\delta}$, then

$$\|T_a u\|_{L^{1,\infty}_\omega} \le C\|u\|_{L^1_\omega} \text{ and } \|T^*_a u\|_{L^{1,\infty}_\omega} \le C\|u\|_{L^1_\omega}.$$

Theorem 1.11. *Let $0 < \varrho < 1$. For any $1 < p < \infty$ and $\omega \in A_p$, if $a \in L^\infty S^{-n(1-\varrho)}_\varrho$, then*

$$\|T^*_a u\|_{L^p_\omega} \le C\|u\|_{L^p_\omega}.$$

For $p = 1$ and $\omega \in A_1$, if $a \in L^\infty S^{-n(1-\varrho)}_\varrho$, then

$$\|T^*_a u\|_{L^{1,\infty}_\omega} \le C\|u\|_{L^1_\omega}.$$

The main contribution of these theorems, besides getting the weighted L^p boundedness of T^*_a, is extending the range of ϱ, δ to general case. Especially, the case $p = r = 1$ is considered as well. Here, we would like to highlight potential directions for further research, such as extending the study from L^p spaces to Morrey spaces. For progress on Calderón-Zygmund operators (a class of the PDOs) in Morrey spaces, we refer the reader to [10, 15, 16, 17] and the references therein.

Another topic of this paper is to investigate some properties of pseudo-differential operator T_a and its dual operators T^*_a acting on Hardy spaces $H^p(\mathbb{R}^n)$, where $0 < p \le 1$. The first property is (H^p, H^p) continuity, which can go back to the studies [2, 3]. They introduce strongly singular Calderón-Zygmund operators T and prove the operators T satisfying $T^*(1) = 0$ acts continuously on $H^p(\mathbb{R}^n)$ for $p_0 < p \le 1$. As an application, they point out that the pseudo-differential operators T_a with symbols in $S^{-\frac{n}{2}(1-\varrho)}_{\varrho,\delta}$ are included in strongly singular Calderón-Zygmund operators, where $0 < \delta \le \varrho < 1$. Later, in [1] the authors extend the range of ϱ and δ to more general case, that is, $0 < \varrho \le 1$ and $0 \le \delta < 1$, but $a \in S^{-\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$.

Theorem 1.12 (See [1])**.** *Let $0 < \varrho \le 1$, $0 \le \delta < 1$ and $a(x,\xi) \in S^m_{\varrho,\delta}$. If*

$$m \le -\frac{n}{2}(1-\varrho) - \frac{n}{2}\max\{\delta - \varrho, 0\}$$

and $T^*_a(1) = 0$ in the sense of BMO. Then, T_a maps continuously H^p into itself for $p_0 < p \le 1$, where $\frac{1}{p_0} = \frac{1}{2} + \frac{\frac{n}{2}(1-\varrho)(1/\varrho+n/2)}{n(1/\varrho-1+\frac{n}{2}(1-\varrho))}$.

The approach to prove this theorem is applying the atomic and molecular characterization of $H^p(\mathbb{R}^m)$. The advantage of this approach is that one only needs to show that $T_a a_Q$, the image of a $(p,2)$ atom a_Q, is a suitable molecule. The condition that $T_a^*(1)=0$ is used only to provide the cancellation condition of the molecule, that is, $\int_{\mathbb{R}^n} T_a a_Q(x)dx=0$, at cost of restricting the range of p into $p_0<p\le 1$. So, the higher degree of cancellation, namely,

$$T_a^*(x^\alpha)=0, \quad for \quad |\alpha|\le [n(\frac{1}{p}-1)], \tag{6}$$

is required to extend for p below p_0. Here and below, $[x]$ indicates the integer part of x. See [13, 18, 37] for the case of Calderón-Zygmund operators. Notice that (6) is used only to provide $\int_{\mathbb{R}^n} x^\alpha T_a a_Q(x)dx=0$ for $|\alpha|\le[n(\frac{1}{p}-1)]$. So, we use the following condition instead of (6) in this paper as:

Definition 1.1. *Let $0<p\le 1$, $t\in\mathbb{N}^+\cup\{0\}$, T be a operator and $L^2_{c,t}(\mathbb{R}^n)$ denote the set of functions in $L^2_c(\mathbb{R}^n)$ such that$\int_{\mathbb{R}^n} x^\beta f(x)dx=0$ for $|\beta|\le t$. If $f\in L^2_{c,t}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} x^\alpha Tf(x)dx=0, \quad for \quad |\alpha|\le[n(\frac{1}{p}-1)]. \tag{7}$$

Here, $L^2_c(\mathbb{R}^n)$ denotes the set of functions in $L^2(\mathbb{R}^n)$ with compact support.

As we known, for the atomic decomposition of an element of $H^p(\mathbb{R}^n)$, one can always choose $(p,2)$ atoms with an number of additional vanishing moments that is known as $(p,2,t)$ atoms with $t\ge[n(\frac{1}{p}-1)]$ (see [35]). Clearly, if f is a $(p,2,t)$ atom, then $f\in L^2_{c,t}(\mathbb{R}^n)$ with $t\ge[n(\frac{1}{p}-1)]$. Moreover, the proof of Proposition 3.1 below implies that (7) for both T_a and T_a^* are well defined, where the symbol a is given in Theorem 1.13.

Theorem 1.13. *Let $0<p<1$, $0\le\varrho\le 1$ and $0\le\delta<1$.*

1. *If T_a^* defined as (2) satisfies condition (7) and $a\in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}$. Then, the operator T_a^* is bounded on $H^p(\mathbb{R}^n)$.*
2. *If T_a defined as (1) satisfies condition (7) and $a\in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)}$. Then, the operator T_a is bounded on $H^p(\mathbb{R}^n)$.*

Compared with Theorem 1.12, Theorem 1.13 extend p below p_0 and improve the range of m.

The second property investigated in this paper is (H^p,L^p) continuity of pseudo-differential operators, which can go back to the results of the papers [11] and [9] for $p=1$, which is extended to the case $0<p\le 1$ as given in [33].

Theorem 1.14 (See [33])**.** *Let $0<p\le 1$ and $0\le\delta\le\varrho<1$. If $a\in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}$. Then, the operators T_a defined as (1) is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Actually, the authors of the paper [33] get that T_a is continuously h_p into itself. Here, h_p denotes the local Hardy spaces introduced in [14]. We also refer to [30, 31] for the extension to Triebel-Lizorkin spaces that coincident with the local Hardy spaces for some special index. Theorem 1.14 holds because of the fact $H^p\subset h^p\subset L^p$ for $0<p<\infty$. As we see, the case $0\le\varrho<\delta<1$ is not considered in Theorem 1.14. And this case is considered in [1] later.

Theorem 1.15 (In [1])**.** *Let $0<\varrho\le 1$, $0\le\delta<1$ and p_0 given as Theorem 1.12 (it is understood that for $\varrho=1$, $p_0=n/(n+1)$). If $a\in S_{\varrho,\delta}^{-\frac{n}{2}(1-\varrho)-\frac{n}{2}\max(0,\delta-\varrho)}$. Then, the operators T_a defined as in (1) is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p_0\le p\le 1$, when $0<\varrho<1$, and for $p_0<p\le 1$, when $\varrho=1$.*

Compared with Theorem 1.14, Theorem 1.15 relaxes the range of ϱ, δ, but put a restriction on p and the order of T_a. Both of them do not contain the case $\varrho = 0$, $0 < \delta < 1$. In this paper, we prove the following result.

Theorem 1.16. *Let $0 < p \leq 1$, $0 \leq \varrho \leq 1$ and $0 \leq \delta < 1$.*

1. *If $a \in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}$. Then, the operators T_a^* defined as (2) is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*
2. *If $a \in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)}$. Then, the operators T_a defined as (1) is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

2. The proof of pointwise estimate for the sharp maximal function

Let us denote

$$K(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,\xi)d\xi \text{ and } K^*(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(y,\xi)d\xi. \tag{8}$$

Then, T_a and T_a^* can be written as:

$$T_a u(x) = \int_{\mathbb{R}^n} K(x,y)u(y)dy \quad \text{and} \quad T_a^* u(x) = \int_{\mathbb{R}^n} K^*(x,y)u(y)dy \tag{9}$$

respectively. Now, we introduce the standard Littlewood-Paley partition of unity. Let $C > 1$ be a constant. Set $E_{-1} = \{\xi : |\xi| \leq 2C\}$, $E_j = \{\xi : C^{-1}2^j \geq |\xi| \leq C2^{j+1}\}$, $j = 0, 1, 2, \cdots$.

Lemma 2.1. *There exist $\psi_{-1}(\xi), \psi(\xi) \in C_0^\infty$, such that*

1. *$supp\ \psi \subset E_0$, $supp\ \psi_{-1} \subset E_{-1}$;*
2. *$0 \leq \psi \leq 1$, $0 \leq \psi_{-1} \leq 1$;*
3. *$\psi_{-1}(\xi) + \sum_{j=1}^{\infty} \psi(2^{-j}\xi) = 1$.*

By Lemma 2.1, the symbol $a(x,\xi)$ can been written as:

$$a(x,\xi) = a(x,\xi)\big(\psi_{-1}(\xi) + \sum_{j=1}^{\infty}\psi(2^{-j}\xi)\big) =: \sum_{j=0}^{\infty} a_j(x,\xi).$$

Consequently, the operator T_a and T_a^* can been decomposed as:

$$T_a u(x) = \sum_{j=0}^{\infty} T_j u(x) \text{ and } T_a^* u(x) = \sum_{j=0}^{\infty} T_j^* u(x), \tag{10}$$

respectively, where

$$T_j u(x) = \int_{\mathbb{R}^n} K_j(x,y)u(y)dy \quad \text{with} \quad K_j(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a_j(x,\xi)d\xi, \tag{11}$$

$$T_j^* u(x) = \int_{\mathbb{R}^n} K_j^*(x,y)u(y)dy \quad \text{with} \quad K_j^*(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a_j(y,\xi)d\xi. \tag{12}$$

Lemma 2.2. *Let $0 \leq \varrho \leq 1$, $0 \leq \delta < 1$ and $a(x,\xi) \in S_{\varrho,\delta}^m$. If $1 < p \leq 2 \leq q < \infty$ and*

$$m \leq -n(\frac{1}{p} - \frac{1}{q}) - \frac{n}{2}\max\{\delta - \varrho, 0\},$$

then

$$\|T_a u\|_{L^q} \lesssim \|u\|_{L^p} \text{ and } \|T_a^* u\|_{L^q} \lesssim \|u\|_{L^p}.$$

By Hardy-Littlewood-Sobolev estimate and L^2-estimate for pseudo-differential operators, in [1] the authors proved the first inequality in the case of $0 < \varrho \leq 1$. The case $\varrho = 0$ and the second inequality can been obtained by the same way.

Lemma 2.3. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $0 \leq \varrho \leq 1$, $0 \leq \delta < 1$ and $1 < p \leq 2$. For any positive integer j satisfying $2^j l < 1$, if $a(x,\xi) \in S^{-\frac{n}{p}(1-\varrho)}_{\varrho,\delta}$, then*

$$\int_{\mathbb{R}^n} |u(y)||K_j(x,y) - K_j(z,y)|dy \lesssim 2^j l M_p u(x_0), \quad \forall x, z \in Q(x_0, l). \tag{13}$$

if $a(x,\xi) \in S^{-\frac{n}{p}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$, then

$$\int_{\mathbb{R}^n} |u(y)||K^*_j(x,y) - K^*_j(z,y)|dy \lesssim 2^j l M_p u(x_0), \quad \forall x, z \in Q(x_0, l). \tag{14}$$

Proof. The idea behind the proof of (13) is standard which could be found in [8]. So, we omit it here. However, to prove (14), this method has to be modified since the Parseval's identity can not be used directly. Firstly, the integrand of left side of (14) can be bounded by:

$$\int_{\mathbb{R}^n} |u(y)|| \int_{\mathbb{R}^n} \left(e^{i\langle x-y,\xi\rangle} - e^{i\langle z-y,\xi\rangle}\right) a_j(y,\xi) d\xi | dy. \tag{15}$$

Break up this integrand as follows

$$\int_{|y-x_0|\leq 2^{-j\varrho+1}} + \int_{|y-x_0|>2^{-j\varrho+1}}$$

Hölder's inequality show that the first term is bounded by:

$$\left(\int_{|y-x_0|\leq 2^{-j\varrho+1}} |u(y)|^p dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} | \int_{\mathbb{R}^n} e^{i\langle \tilde{x}-y,\xi\rangle} a_j(y,\xi)(x-z)\cdot\xi d\xi|^{p'} dy\right)^{\frac{1}{p'}}, \tag{16}$$

where $\tilde{x}$ denotes some point between x and z. For any fixed x and z, let $b_j(y,\xi) = a_j(\tilde{x} - y,\xi)|\xi|^{\frac{n}{p}(1-\varrho)-n(\frac{1}{2}-\frac{1}{p'})}$ and $\widehat{g_j(\xi)} = |\xi|^{-\frac{n}{p}(1-\varrho)+n(\frac{1}{2}-\frac{1}{p'})}\chi_j(\xi)(x-z)\cdot\xi$. Then, we can write

$$\int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} a_j(\tilde{x}-y,\xi)(x-z)\cdot\xi d\xi = T_{b_j} g_j(y).$$

Notice that $b_j \in S^{-n(\frac{1}{2}-\frac{1}{p'})-\frac{n}{2}\max\{\delta-\varrho,0\}}_{\varrho,\delta}$, by Lemma 2.2 we have

$$\|T^*_{b_j} g_j\|_{L^{p'}} \lesssim \|g_j\|_{L^2} = \|\hat{g}_j\|_{L^2}.$$

Therefore, the formula given in (16) is bounded by

$$2^j l M_p u(x_0).$$

By Hölder's inequality, integrating by parts and the fact that $|y - x_0| \sim |y - x|$ follows from $2^j l < 1$, $x \in Q(x_0, l)$ and $|y - x_0| > 2^{-j\varrho+1}$, the second term is bounded by:

$$\left(\int_{|y-x_0|>2^{-j\varrho+1}} \frac{|u(y)|^p}{|y-x_0|^{pN}} dy\right)^{\frac{1}{p}} \times \sum_{|\alpha|=N} \left(\int_{\mathbb{R}^n} | \int_{\mathbb{R}^n} e^{i\langle \tilde{x}-y,\xi\rangle} \partial^\alpha_\xi \left(a_j(y,\xi)(x-z)\cdot\xi\right) d\xi|^{p'} dy\right)^{\frac{1}{p'}}. \tag{17}$$

For any fixed x and z, let $\tilde{b}_j(y,\xi) = \partial_\xi^\alpha\big(a_j(y,\xi)(x-z)\cdot\xi\big)|\xi|^{\frac{n}{p}(1-\varrho)-n(\frac{1}{2}-\frac{1}{p'})+\varrho|\alpha|}$ and $\widehat{\tilde{g}_j(\xi)} = |\xi|^{-\frac{n}{p}(1-\varrho)+n(\frac{1}{2}-\frac{1}{p'})-\varrho|\alpha|}\chi_j(\xi)$. Then, we can write

$$\int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle}\partial_\xi^\alpha\big(a_j(\tilde{x}-y,\xi)(x-z)\cdot\xi\big)d\xi = T^*_{\tilde{b}_j}\tilde{g}_j(y).$$

Clearly, $\tilde{b}_j \in S_{\varrho,\delta}^{-n(\frac{1}{2}-\frac{1}{p'})-\frac{n}{2}\max\{\delta-\varrho,0\}}$ with bounds $\lesssim 2^j l$. Moreover, by Lemma 2.2 we have

$$\|T^*_{\tilde{b}_j}\tilde{g}_j\|_{L^{p'}} \lesssim 2^j l\|\tilde{g}_j\|_{L^2} = 2^j l\|\hat{\tilde{g}}_j\|_{L^2}.$$

By a simple calculation, we can get the expression given in (17) is bounded by

$$\lesssim \quad 2^j l M_p u(x_0).$$

Thus, the following desired estimate can be provided. □

Lemma 2.4. *Let $Q(x_0,l)$ be a fixed cube with side length $l < 1$ $0 \le \varrho \le 1$ and $0 \le \delta < 1$. For any positive integer $N > \frac{n}{p}$ and any positive integer j with $l^{-1} \le 2^j \le l^{-\frac{1}{\varrho}}$, if $a \in S_{\varrho,\delta}^{-\frac{n}{p}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}$, then*

$$\frac{1}{|Q|}\int_{Q(x_0,l)}|T_j u(x)|dx \quad\lesssim\quad 2^{j\frac{n}{2}(\frac{n}{Np}-1)}l^{\frac{n}{2}(\frac{n}{Np}-1)}M_p u(x_0) \tag{18}$$

and

$$\frac{1}{|Q|}\int_{Q(x_0,l)}|T_j^* u(x)|dx \quad\lesssim\quad 2^{j\frac{n}{2}(\frac{n}{Np}-1)}l^{\frac{n}{2}(\frac{n}{Np}-1)}M_p u(x_0). \tag{19}$$

Remark 2.1. *If $\varrho = 0$, the condition $l^{-1} \le 2^j \le l^{-\frac{1}{\varrho}}$ is interpreted as $l^{-1} \le 2^j$. If $\varrho = 1$, this lemma is no use.*

Proof. Notice that $a(x,\xi)\psi(2^{-j}\xi) \in S_{\varrho,\delta}^{-n(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max\{\delta-\varrho,0\}}$ with the bounds $\lesssim 2^{-j\frac{n}{p}(1-\varrho)+n(\frac{1}{p}-\frac{1}{2})}$. So, T_j is bounded from L^p to L^2, see Lemma 2.2. More exactly, we have

$$\|T_j u\|_{L^2} \lesssim 2^{-j\frac{n}{p}(1-\varrho)+n(\frac{1}{p}-\frac{1}{2})}\|u\|_{L^p}.$$

Let integral N defined as above and set

$$T = l^{\frac{n}{2N}}2^{j(\frac{n}{2N}-\varrho)},$$

$$u_1(x) = u(x)\chi_{Q(x_0,4T)}(x) \quad \text{and} \quad u_2(x) = u(x) - u_1(x), \tag{20}$$

where $\chi_{Q(x_0,4T)}(x)$ is the characteristic function of the ball $Q(x_0,4T)$. Then, the left hand of (18) can be bounded by:

$$\int_{Q(x_0,l)}|T_j u_1(x)|dx + \int_{Q(x_0,l)}|T_j u_2(x)|dx =: M_1 + M_2.$$

Hölder's inequality and $(p,2)$-boundedness of T_j imply that M_1 is bounded by:

$$\begin{aligned} l^{\frac{n}{2}}\|T_j u_1\|_{L^2} &\lesssim 2^{-j\frac{n}{p}(1-\varrho)+n(\frac{1}{p}-\frac{1}{2})}l^{\frac{n}{2}}\|u_1\|_{L^p} \\ &\lesssim 2^{j\frac{n}{2}(\frac{n}{Np}-1)}l^{\frac{n}{2}(\frac{n}{Np}+1)}M_p u(x_0). \end{aligned} \tag{21}$$

For M_2, noticing that for any $x \in Q(x_0,l)$ and any $y \in Q^C(x_0,4T)$, we have

$$|y-x| \ge \frac{|y-x_0|}{2}.$$

Hölder's inequality, integrating by parts and Parseval's identity give that $|T_j u_2(x)|$ is bounded by

$$\begin{aligned}
&\left(\int_{|y-x_0|>4T} \frac{|u(y)|^p}{|y-x_0|^{pN}} dy\right)^{\frac{1}{p}} \left(\int_{|y-x_0|>4T} |y-x_0|^{p'N} \Big|\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,\xi)\psi(2^{-j}\xi)d\xi\Big|^{p'} dy\right)^{\frac{1}{p'}}\\
\lesssim\ & \left(\int_{|y-x_0|>4T} \frac{|u(y)|^p}{|y-x_0|^{pN}} dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha a(x,\xi)\psi(2^{-j}\xi)|^p d\xi\right)^{\frac{1}{p}}\\
\lesssim\ & 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n}{2}(\frac{n}{Np}-1)} M_p u(x_0).
\end{aligned}$$

So

$$M_2 = \int_{Q(x_0,l)} |T_j u_2(x)| dx \lesssim 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n}{2}(\frac{n}{Np}+1)} M_p u(x_0). \tag{22}$$

Thus, the desired estimate (18) follows from (21) and (22). So, we complete the proof. □

Lemma 2.5. *Let $Q(x_0,l)$ be a fixed cube with side length $l<1$. Suppose $0<\varrho<\delta<1$, $a \in S^{-\frac{n}{p}(1-\varrho)}_{\varrho,\delta}$. Then for any $1 \le \lambda \le \frac{1}{\varrho}$, any positive integer $N > \frac{n}{p}$ and any positive integer j with $l^{-\lambda} \le 2^j \le l^{-\frac{1}{\varrho}}$, we have*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)| dx \ \lesssim\ \big(2^{j\delta} l^\lambda + 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n\lambda}{2}(\frac{n}{Np}-1)}\big) M_p u(x_0).$$

Proof. If $1<\lambda\le\frac{1}{\varrho}$, then $l^\lambda < l$. Take integer L such that it is the first number no less than $l^{1-\lambda}$, that is $L-1 < l^{1-\lambda} \le L$. Then, there are L^n cubes with the same side length l^λ covering $Q(x_0,l)$. Moreover, we have

$$Q(x_0,l) \subset \cup_{i=1}^{L^n} Q(x_i,l^\lambda) \subset Q(x_0,2l).$$

Clearly, $L^n \le 2^n l^{n(1-\lambda)}$. Denote

$$T_{j,i}u(x) \ = \ \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x_i,\xi)\psi(2^{-j}\xi)\hat{u}(\xi)d\xi. \tag{23}$$

We can write

$$\begin{aligned}
&\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)| dx\\
\le\ & \frac{1}{|Q|}\sum_{i=1}^{L^n}\Big(\int_{Q(x_i,l^\lambda)} |T_j u(x) - T_{j,i}u(x)| dx + \int_{Q(x_i,l^\lambda)} |T_{j,i}u(x)| dx\Big).
\end{aligned} \tag{24}$$

Now, we claim that

$$|T_j u(x) - T_{j,i}u(x)| \lesssim |x-x_i| 2^{j\delta} M_p u(x_0), \tag{25}$$

$$\int_{Q(x_i,l^\lambda)} |T_{j,i}u(x)| dx \ \lesssim\ 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n\lambda}{2}(\frac{n}{Np}+1)} M_p u(x_0). \tag{26}$$

Since $L^n \le 2^n l^{n(1-\lambda)}$, we can get the desired estimate by substituting both (25) and (26) into (24).

Note that $|T_j u(x) - T_{j,i}u(x)|$ is bounded by:

$$\int_{\mathbb{R}^n} |u(y)| \Big|\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\big(a(x,\xi)-a(x_i,\xi)\big)\psi(2^{-j}\xi)d\xi\Big| dy.$$

Then, (25) follows from the same argument as (36).

Now, we will prove (26). For fixed x_i, we can see that $a(x_i,\xi)\psi(2^{-j}\xi) \in S_{\varrho,0}^{-n(\frac{1}{p}-\frac{1}{2})}$ with the bounds $\lesssim 2^{-j\frac{n}{p}(1-\varrho)+n(\frac{1}{p}-\frac{1}{2})}$. So, $T_{j,i}$ is bounded from L^p to L^2. More exactly, we have

$$\|T_{j,i}u\|_{L^2} \lesssim 2^{-j\frac{n}{p}(1-\varrho)+n(\frac{1}{p}-\frac{1}{2})}\|u\|_{L^p}.$$

Fix positive integral N large enough and set

$$T = l^{\frac{n\lambda}{2N}} 2^{j(\frac{n}{2N}-\varrho)},$$

$$u_{i,1}(x) = u(x)\chi_{Q(x_i,4T)}(x) \quad \text{and} \quad u_{i,2}(x) = u(x) - u_{i,1}(x), \tag{27}$$

where $\chi_{Q(x_i,4T)}(x)$ is the characteristic function of the ball $Q(x_i,4T)$. Then, (26) follows from the same argument as (18).

If $\lambda = 1$, we define

$$T_{j,0}u(x) \quad = \quad \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x_0,\xi)\psi(2^{-j}\xi)\hat{u}(\xi)d\xi. \tag{28}$$

Then, the desired estimate can be obtained by the same argument as above with $T_{j,i}u$ replaced by $T_{j,0}u$. So, the proof is completed. $\square$

We remark that the same result holds for the case $\varrho = 0$. Here, the range of λ can be extended to $[1,\infty)$. However, to make some sums convergent, λ has to be confined to a finite range.

Lemma 2.6. *Let $Q(x_0,l)$ be a fixed cube with side length $l < 1$. Suppose $\varrho = 0$, $0 < \delta < 1$, $a \in S_{0,\delta}^{-\frac{n}{p}}$, then for any $1 \le \lambda \le \frac{2}{p(1-\delta)}$, any positive integer $N > \frac{n}{p}$ and any positive integer j with $l^{-\lambda} \le 2^j \le l^{-\frac{2}{p(1-\delta)}}$,*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)|dx \quad \lesssim \quad \big(2^{j\delta}l^{\lambda} + 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n\lambda}{2}(\frac{n}{Np}-1)}\big) M_p u(x_0).$$

Lemma 2.7. *Let $Q(x_0,l)$ be a fixed cube with side length $l < 1$. Suppose $1 < p \le 2$, $\varrho = 0$, $0 \le \delta < 1$, $a \in S_{0,\delta}^{-\frac{n}{p}}$, then for any positive integer $N > \frac{n}{p}$ and any positive integer j with $l^{-\frac{2}{p(1-\delta)}} \le 2^j$,*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)|dx \quad \lesssim \quad 2^{-j\frac{n}{2}(1-\delta)(1-\frac{n}{pN})} l^{-\frac{n}{p}(1-\frac{n}{pN})} M_p u(x_0).$$

Proof. Denote

$$\Gamma = 2^{j\frac{n}{pN}(1-\delta)} l^{\frac{n}{pN}}.$$

Set $u_3(x) = u(x)\chi_{Q(x_0,2\Gamma)}(x)$ and $u_4(x) = u(x) - u_3(x)$. Then

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)|dx \le \frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u_3(x)|dx + \frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u_4(x)|dx. \tag{29}$$

Notice that $a(x,\xi)\psi(2^{-j}\xi) \in S_{\varrho,\delta}^{-n(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\delta}$ with bounds $\lesssim 2^{-j\frac{n}{2}(1-\delta)}$. Hölder's inequality and the L^p-estimate of T_j give that

$$\begin{aligned}\frac{1}{|Q|}\int_{Q} |T_j u_3(x)|dx \quad &\lesssim \quad 2^{-j\frac{n}{2}(1-\delta)} l^{-\frac{n}{p}} \|u_1\|_{L^p} \lesssim 2^{-j\frac{n}{2}(1-\delta)} l^{-\frac{n}{p}} \Gamma^{\frac{n}{p}} M_p u(x_0) \\ &= \quad 2^{-j\frac{n}{2}(1-\delta)(1-\frac{n}{pN})} l^{-\frac{n}{p}(1-\frac{n}{pN})} M_p u(x_0).\end{aligned} \tag{30}$$

Notice that $\Gamma > l$. We have $|y-x| \sim |y-x_0|$ for $\forall x \in Q(x_0, l)$ and $\forall y \in Q^C(x_0, 2\Gamma)$. So, direct computations show that

$$\begin{aligned} |T_j u_4(x)| &\le \int_{|y-x_0|\ge 2\Gamma} |K_j(x, x-y)||u(y)|dy \lesssim \Gamma^{(\frac{n}{p}-N)} M_p u(x_0) \\ &= 2^{-j\frac{n}{2}(1-\delta)(1-\frac{n}{pN})} l^{-\frac{n}{p}(1-\frac{n}{pN})} M_p u(x_0), \end{aligned}$$

which implies that

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u_4(x)|dx \lesssim 2^{-j\frac{n}{2}(1-\delta)(1-\frac{n}{pN})} l^{-\frac{n}{p}(1-\frac{n}{pN})} M_p u(x_0). \tag{31}$$

Clearly, the desired estimate follows from (29), (30) and (31). □

Taking $\Gamma = l$ in the proof Lemma 2.7, we can get a similar result for $\varrho > 0$ with the same argument as above.

Lemma 2.8. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $0 < \varrho \le 1$, $0 \le \delta < 1$ and $1 < p \le 2$. For any positive integer $N > \frac{n}{p}$ and any positive integer j with $l^{-\frac{1}{\varrho}} \le 2^j$, if $a \in S^{-\frac{n}{p}(1-\varrho)}_{\varrho,\delta}$ then*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)|dx \lesssim \left(2^{-j(\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\})} + 2^{-j\varrho(\frac{n}{p}-N)} l^{\frac{n}{p}-N)}\right) M_p u(x_0)$$

and

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u(x)|dx \lesssim \left(2^{-j(\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\})} + 2^{-j\varrho(\frac{n}{p}-N)} l^{\frac{n}{p}-N)}\right) M_p u(x_0);$$

Proof of Theorem 1.5. Without loss of generality, we assume that the symbol $a(x,\xi)$ vanishes for $|\xi| \le 1$. Let $Q = Q(x_0, l)$ denote the cube centered at x_0 with the side length l. For any fixed cube Q, we are going to prove that

$$\frac{1}{|Q|}\int_Q |T_a u(x) - C_Q|dx \le C M_p u(x_0), \tag{32}$$

where $C_Q = \frac{1}{|Q|}\int_Q T_a u(y)dy$. The proof is trivial for $l \ge 1$, we omit it here. We consider on $0 < l < 1$. Note that the left hand of (32) can be controlled by:

$$\frac{1}{|Q|^2}\int_Q\int_Q |T_a u(x) - T_a u(y)|dydx. \tag{33}$$

We compose the operator T_a as (10), then estimate (33) by

$$\sum_{1<2^j\le l^{-1}} \frac{1}{|Q|^2}\int_Q\int_Q |T_j u(x) - T_j u(z)|dzdx \quad + \sum_{l^{-1}<2^j} \frac{2}{|Q|}\int_Q |T_j u(x)|dx. \tag{34}$$

Lemma 2.3 implies that

$$|T_j u(x) - T_j u(z)| \le \int_{\mathbb{R}^n} |u(y)||K_j(x,y) - K_j(z,y)|dy \le C2^j|x-z|M_p u(x_0).$$

So, the first term in (34) is bounded by

$$M_p u(x_0) l \sum_{1<2^j\le l^{-1}} 2^j \lesssim M_p u(x_0). \tag{35}$$

Next, we claim that the second term in (34) can be controlled by $M_p u(x_0)$ as well.

Case 1. $0 \leq \delta \leq \varrho \leq 1$, $\delta \neq 1$ If $\varrho = 0$, then by Lemma 2.4 and Remark 2.1 the second term in (34) can be bounded by:

$$\sum_{l^{-1}<2^j} 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n}{2}(\frac{n}{Np}-1)} M_p u(x_0) \lesssim M_p u(x_0).$$

If $\varrho \neq 0$, we break up this sum as follows

$$\sum_{l^{-1}<2^j \leq l^{-\frac{1}{\varrho}}} \frac{2}{|Q|} \int_Q |T_j u(x)| dx + \sum_{l^{-\frac{1}{\varrho}}<2^j} \frac{2}{|Q|} \int_Q |T_j u(x)| dx. \tag{36}$$

Then, Lemma 2.4 and Lemma 2.8 imply that they can be controlled by

$$\sum_{l^{-1}<2^j \leq l^{-\frac{1}{\varrho}}} 2^{j\frac{n}{2}(\frac{n}{Np}-1)} l^{\frac{n}{2}(\frac{n}{Np}-1)} M_p u(x_0)$$
$$+ \sum_{l^{-\frac{1}{\varrho}}<2^j} \left(2^{-j(\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\})} + 2^{-j\varrho(\frac{n}{p}-N)} l^{\frac{n}{p}-N}\right) M_p u(x_0) \lesssim M_p u(x_0).$$

Case 2. $0 \leq \varrho < \delta < 1$ If $\varrho \neq 0$, we break up this sum as (36) as well. By Lemma 2.8, the second term in (36) can be controlled by $M_p u(x_0)$. For the first term in (36), we write

$$\begin{aligned}\sum_{l^{-1}<2^j \leq l^{-\frac{1}{\varrho}}} \frac{2}{|Q|} \int_Q |T_j u(x)| dx &= \Big(\sum_{l^{-1}<2^j \leq l^{-\frac{1}{\delta}}} + \sum_{l^{-\frac{1}{\delta}}<2^j \leq l^{-\frac{1}{\delta^2}}} + \ldots + \sum_{l^{-\frac{1}{\delta^{k-1}}}<2^j \leq l^{-\frac{1}{\delta^k}}} \\ &+ \ldots + \sum_{l^{-\frac{1}{\delta^{\gamma-1}}}<2^j \leq \min\{l^{-\frac{1}{\varrho}}, l^{-\frac{1}{\delta^\gamma}}\}} \Big) \frac{1}{|Q|} \int_{Q(x_0,l)} |T_j u(x)| dx,\end{aligned}$$

where γ is the first positive integer such that $\frac{1}{\delta^\gamma} \geq \frac{1}{\varrho}$. Then, take $\lambda = \frac{1}{\delta^k}$, $k = 0, 1, ..., \gamma - 1$ in Lemma 2.5 respectively, we can see that each sum above is bounded by $M_p u(x_0)$. Therefore, we have

$$\sum_{l^{-1}<2^j \leq l^{-\frac{1}{\varrho}}} \frac{2}{|Q|} \int_Q |T_j u(x)| dx \leq C_\gamma M_p u(x_0). \tag{37}$$

If $\varrho = 0$, we break up this sum as follows

$$\sum_{l^{-1}<2^j \leq l^{-\frac{2}{p(1-\delta)}}} \frac{2}{|Q|} \int_Q |T_j u(x)| dx + \sum_{l^{-\frac{2}{p(1-\delta)}}<2^j} \frac{2}{|Q|} \int_Q |T_j u(x)| dx. \tag{38}$$

Applying Lemma 2.6 and Lemma 2.7 instead of Lemma 2.8 and Lemma 2.5, we can get the desired estimate by the same argument as above. So, the proof is finished.

□

Next, we started to prepare for proving the case $p = 1$, that is, Theorem 1.6 and Theorem 1.7.

Lemma 2.9. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $0 \leq \varrho \leq 1$ and $0 \leq \delta < 1$. For any positive integer j satisfying $2^j l < 1$, if $a(x, \xi) \in S^{-n(1-\varrho)}_{\varrho,\delta}$, then*

$$\int_{\mathbb{R}^n} |u(y)||K_j(x, y) - K_j(z, y)| dy \lesssim 2^j l M u(x_0), \quad \forall x, z \in Q(x_0, l) \tag{39}$$

and

$$\int_{\mathbb{R}^n} |u(y)||K_j^*(x, y) - K_j^*(z, y)| dy \lesssim 2^j l M u(x_0), \quad \forall x, z \in Q(x_0, l). \tag{40}$$

Proof. The proof of (39) and (40) is standard. We only show a outline of proving (40). The integrand of left side of (40) can be bounded by:

$$\int_{\mathbb{R}^n} |u(y)|| \int_{\mathbb{R}^n} \left(e^{i\langle x-y,\xi\rangle} - e^{i\langle z-y,\xi\rangle}\right) a_j(y,\xi) d\xi | dy. \tag{41}$$

Break up this integrand as follows

$$\int_{|y-x_0|\leq 2^{-j\varrho+1}} + \int_{|y-x_0|>2^{-j\varrho+1}}.$$

A direct calculation gives the first term is bounded by $2^j l M u(x_0)$, and integration by parts with respect to the variable ξ yields that the second term has the same bound. Thus, the proof is completed. □

Remark 2.2. *Notice that the smooth of variable y in $a(y,\xi)$ is not used in the proof of (40). So, it can be get in a relaxed condition. More exactly, (40) can been obtained under condition $a(y,\xi) \in L^\infty S_\varrho^{-n(1-\varrho)}$.*

Lemma 2.10. *Let $Q(x_0,l)$ be a fixed cube with side length $l < 1$. Suppose $0 < \varrho \leq 1$ and $0 \leq \delta < 1$. For any positive integer j with $l^{-1} \leq 2^j \leq l^{-\frac{1}{\varrho}}$, if $a \in S_{\varrho,\delta}^{-n(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\}}$ then*

$$\frac{1}{|Q(x_0,l)|} \int_{Q(x_0,l)} |T_j f(x)| dx \lesssim 2^{-j\frac{n}{2}} l^{-\frac{n}{2}} M f(x_0); \tag{42}$$

if $a \in S_{\varrho,\delta}^{-n(1-\varrho)}$ then

$$\frac{1}{|Q(x_0,l)|} \int_{Q(x_0,l)} |T_j^* f(x)| dx \lesssim 2^{-j\frac{n}{2}} l^{-\frac{n}{2}} M f(x_0). \tag{43}$$

Proof. Firstly, we will prove (42). Hölder's inequality and Minkowski's inequality implies that the left hand in (42) can be bounded by:

$$l^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left(\int_{|x-x_0|<l} |K_j(x,y)|^2 dx \right)^{\frac{1}{2}} |f(y)| dy.$$

So, it suffices to show

$$\int_{\mathbb{R}^n} \left(\int_{|x-x_0|<l} |K_j(x,y)|^2 dx \right)^{\frac{1}{2}} |f(y)| dy \lesssim 2^{-j\frac{n}{2}} M f(x_0).$$

Break up the integral with respect to the variable y as follows

$$\int_{|y-x_0|\leq 2^{-j\varrho+1}} + \int_{|y-x_0|>2^{-j\varrho+1}}. \tag{44}$$

Let $c_j(x,\xi) = a_j(x,\xi)|\xi|^{n(1-\varrho)}$ and $\widehat{h_j(\xi)} = |\xi|^{-n(1-\varrho)}\chi_j(\xi)$. Then, we can write

$$K_j(x,y) = \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a_j(x,\xi) d\xi = \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} c_j(x,\xi) \hat{h_j}(\xi) d\xi = T_{c_j} h_j(x-y).$$

So, the first term in (44) can be written as

$$\int_{|y-x_0|\leq 2^{-j\varrho+1}} \big(\int_{\mathbb{R}^n} |T_{c_j} h_j(x-y)|^2 dx \big)^{\frac{1}{2}} |f(y)| dy.$$

Notice $c_j \in S_{\varrho,\delta}^{-\frac{n}{2}\max\{\delta-\varrho,0\}}$. Moreover, T_{c_j} is bounded on L^2. So, it can be bounded by

$$\int_{|y-x_0|\leq 2^{-j\varrho+1}} |f(y)| dy \big(\int_{\mathbb{R}^n} |h_j(\xi)|^2 d\xi \big)^{\frac{1}{2}} \leq 2^{-j\frac{n}{2}} M f(x_0).$$

Now, we will estimate the second term in (44). For positive integer $N > n$, denote $\tilde{c}_j(x,\xi) = \partial_\xi^N a_j(x,\xi)|\xi|^{n(1-\varrho)+\varrho N}$ and $\widehat{\tilde{h}_j(\xi)} = |\xi|^{-n(1-\varrho)-\varrho N}\chi_j(\xi)$. Then, we can write

$$K_j(x,y) = \frac{1}{|x-y|^N}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\partial_\xi^N a_j(x,\xi)d\xi = \frac{1}{|x-y|^N}T_{\tilde{c}_j}\tilde{h}_j(x-y).$$

So, the second term in (44) can be written as

$$\int_{|y-x_0|>2^{-j\varrho+1}}\Big(\int_{|x-x_0|<l}|\frac{1}{|y-x|^N}T_{\tilde{c}_j}\tilde{h}_j(x-y)|^2dx\Big)^{\frac{1}{2}}|f(y)|dy.$$

Notice that $|y-x|\sim|y-x_0|$ for any $|x-x_0|<l$ and $|y-x_0|>2^{-j\varrho+1}\geq 2l$. Then, it is bounded by

$$\int_{|y-x_0|>2^{-j\varrho+1}}\frac{1}{|y-x_0|^N}\Big(\int_{\mathbb{R}^n}|T_{\tilde{c}_j}\tilde{h}_j(x-y)|^2dx\Big)^{\frac{1}{2}}|f(y)|dy.$$

Clearly, $\tilde{c}_j\in S_{\varrho,\delta}^{-\frac{n}{2}\max\{\delta-\varrho,0\}}$. So, L^2 boundedness of $T_{\tilde{c}_j}$ gives that it has bound

$$\int_{|y-x_0|>2^{-j\varrho+1}}\frac{1}{|y-x_0|^N}|f(y)|dy\Big(\int_{\mathbb{R}^n}|\tilde{h}_j(\xi)|^2d\xi\Big)^{\frac{1}{2}}\leq 2^{-j\frac{n}{2}}Mf(x_0).$$

For (43), it can be provided by the same argument as above with L^2 boundedness of pseudo-differential operators replaced by Parseval's identity. So, the proof is completed. □

We remark that there is no use for the smoothness of variable y of $a(y,\xi)$ when we prove (43). So, the condition on $a(y,\xi)$ can be relaxed. More exactly, have the following results

Lemma 2.11. *Let $Q(x_0,l)$ be a fixed cube with side length $l<1$. Suppose $0<\varrho\leq 1$. For any positive integer j with $l^{-1}\leq 2^j\leq l^{-\frac{1}{\varrho}}$, if $a\in L^\infty S_\varrho^{-n(1-\varrho)}$ then*

$$\frac{1}{|Q(x_0,l)|}\int_{Q(x_0,l)}|T_j^*f(x)|dx \lesssim 2^{-j\frac{n}{2}}l^{-\frac{n}{2}}Mf(x_0). \tag{45}$$

Lemma 2.12. *Let $Q(x_0,l)$ be a fixed cube with side length $l<1$. Suppose $0<\varrho<\delta<1$. For for any $1\leq\lambda\leq\frac{1}{\varrho}$ and any positive integer j with $l^{-\lambda}\leq 2^j\leq l^{-\frac{1}{\varrho}}$, if $a\in S_{\varrho,\delta}^{-n(1-\varrho)}$ then*

$$\frac{1}{|Q(x_0,l)|}\int_{Q(x_0,l)}|T_jf(x)|dx \lesssim \big(l^\lambda 2^{j\delta}+l^{-\frac{n\lambda}{2}}2^{-j\frac{n}{2}}\big)Mf(x_0).$$

Proof. The proof can be completed by a similar argument as in the proof of Lemma 2.5. Using the notations in them, one can write

$$\begin{aligned}&\frac{1}{|Q|}\int_{Q(x_0,l)}|T_jf(x)|dx\\ \leq\ &\frac{1}{|Q|}\sum_{i=1}^{L^n}\Big(\int_{Q(x_i,l^\lambda)}|T_jf(x)-T_{j,i}f(x)|dx+\int_{Q(x_i,l^\lambda)}|T_{j,i}f(x)|dx\Big).\end{aligned}$$

It is easy to get

$$|T_jf(x)-T_{j,i}f(x)|\lesssim l^\lambda 2^{j\delta}Mf(x_0)$$

and

$$\int_{Q(x_i,l^\lambda)}|T_{j,i}f(x)|dx \lesssim 2^{-j\frac{n}{2}}l^{\frac{n\lambda}{2}}Mf(x_0).$$

Recall $L^n\leq 2^n l^{n(1-\lambda)}$, the desired estimate can be obtained immediately. □

Applying weak $(1,1)$ estimate for T_j and Kolmogorov's inequality instead of L^p estimate in the proof Lemma 2.8, we can get a similar result for $\varrho > 0$.

Lemma 2.13. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $0 < \varrho \le 1$, $0 \le \delta < 1$ and $0 < \epsilon < 1$. For any positive integer $N > n$ and any positive integer j with $l^{-\frac{1}{\varrho}} \le 2^j$, if $a \in S^{-n(1-\varrho)}_{\varrho,\delta}$ then*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j u(x)|^\epsilon dx \lesssim \big(2^{-j(\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\})\epsilon} + 2^{-j\varrho(n-N)\epsilon} l^{n-N)\epsilon}\big)\big(Mu(x_0)\big)^\epsilon \tag{46}$$

and

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u(x)|^\epsilon dx \lesssim \big(2^{-j(\frac{n}{2}(1-\varrho)-\frac{n}{2}\max\{\delta-\varrho,0\})\epsilon} + 2^{-j\varrho(n-N)\epsilon} l^{n-N)\epsilon}\big)\big(Mu(x_0)\big)^\epsilon. \tag{47}$$

Proof of Theorem 1.6. Without loss of generality, we assume that the symbol $a(x,\xi)$ vanishes for $|\xi| \le 1$. Let $Q = Q(x_0, l)$ denote the cube centered at x_0 with the side length l. For any fixed cube Q, we are going to prove that

$$\frac{1}{|Q|}\int_Q \big||T_a u(x)|^\epsilon - |C_Q|^\epsilon\big| dx \lesssim \big(Mu(x_0)\big)^\epsilon,$$

where $C_Q = \frac{1}{|Q|}\int_Q T_a u(y)dy$. Notice that $||a|^\epsilon - |b|^\epsilon| \le |a-b|^\epsilon$, $0 < \epsilon < 1$, it suffices to prove

$$\frac{1}{|Q|}\int_Q |T_a u(x) - C_Q|^\epsilon dx \lesssim \big(Mu(x_0)\big)^\epsilon. \tag{48}$$

Clearly, the left hand integral in (48) for any $0 < \epsilon < 1$ can be bounded by

$$\sum_j \frac{2}{|Q|^2}\int_Q\int_Q |T_j u(x) - T_j u(y)|^\epsilon dydx.$$

Then, by the same argument as the proof Theorem 1.6, we can get the desired estimate. □

Proof of Theorem 1.7. We give a outline here, since it can be proved by a similar argument as above. Clearly, it suffices to show

$$\sum_j \frac{2}{|Q|^2}\int_Q\int_Q |T_j^* u(x) - T_j^* u(y)| dydx \lesssim Mu(x_0).$$

For the case $l < 1$. Break up this sum as follows

$$\sum_{2^j < l^{-1}} + \sum_{l^{-1} \le 2^j \le l^{-\frac{1}{\varrho}}} + \sum_{l^{-\frac{1}{\varrho}} < 2^j}.$$

Then, we can get the desired estimate for the first term ($2^j < l^{-1}$) and the second term ($l^{-1} \le 2^j \le l^{-\frac{1}{\varrho}}$) by Remark 2.2 and Lemma 2.11 respectively. As for the last term ($l^{-\frac{1}{\varrho}} < 2^j$) and the case $l > 1$, it can be estimated by following lemma. So, the proof is completed. □

Lemma 2.14. *Suppose $0 < \varrho < 1$. For any positive integer $N > n$, if $a \in L^\infty S_\varrho^{-n(1-\varrho)}$ then for $0 < \theta < \frac{n}{2}(1-\varrho)$*

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u(x)| dx \lesssim \big(2^{-j(\frac{n}{2}(1-\varrho)-\theta)} + 2^{-j\varrho(n-N)} l^{n-N)}\big) Mu(x_0). \tag{49}$$

Proof. We show an outline here. Set $u_5(x) = u(x)\chi_{Q(x_0,2l)}(x)$ and $u_6(x) = u(x) - u_5(x)$. Then

$$\frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u(x)|dx \leq \frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u_5(x)|dx + \frac{1}{|Q|}\int_{Q(x_0,l)} |T_j^* u_6(x)|dx. \tag{50}$$

Notice that $a_j(y,\xi) \in S_{\varrho,\delta}^{-\frac{n}{2}(1-\varrho)-\theta}$ with bounds $\lesssim 2^{-j\frac{n}{2}(1-\varrho)+j\theta}$. The L^1-estimate of T_j^* given that

$$\frac{1}{|Q|}\int_{Q} |T_j^* u_5(x)|dx \quad \lesssim \quad 2^{-j\frac{n}{2}(1-\varrho)+j\theta} Mu(x_0). \tag{51}$$

Notice that $|y-x| \sim |y-x_0|$ for $\forall x \in Q(x_0,l)$ and $\forall y \in Q^C(x_0,2l)$. So, integrating by parts gives that

$$|T_j^* u_6(x)| \lesssim 2^{-j\varrho(n-N)} l^{n-N)} Mu(x_0). \tag{52}$$

Clearly, the desired estimate follows from (50), (51) and (52). □

3. The proof of continuity on Hardy spaces

A tempered distribution f belongs to Hardy spaces $H^p(\mathbb{R}^n)$ if, for some $\phi \in \Im$ with $\int_{\mathbb{R}^n}\phi(x)dx \neq 0$, the maximal operator

$$M_\phi f(x) := \sup_{t>0} |f * \phi_t(x)|$$

is in $L^p(\mathbb{R}^n)$, where $\phi_t(x) = t^{-n}\phi(x/t)$. The continuity properties of pseudo-differential operator T_a and operators T_a^* acting on Hardy spaces $H^p(\mathbb{R}^n)$ will be done by standard atomic and molecular technique [36].

Definition 3.1. *Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, and the nonnegative integer $s \geq [n(\frac{1}{p}-1)]$. A function $a(x) \in L^q(\mathbb{R}^n)$ is called a (p,q,s) atom with the center at x_0, if it satisfies the following conditions:*

$$(1)\ supp a_Q \subset Q; \quad (2)\ \int_{\mathbb{R}^n} |a_Q(y)|^q \leq |Q|^{1-\frac{q}{p}}; \quad (3)\ \int_{\mathbb{R}^n} a_Q(y)y^\alpha dy = 0, 0 \leq |\alpha| \leq s.$$

Definition 3.2. *(See [36]) Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, and the nonnegative integer $s \geq [n(\frac{1}{p}-1)]$, $\epsilon > \max\{\frac{s}{n}, \frac{1}{p}-1\}$,$a_0 = 1-\frac{1}{p}+\epsilon$ and $b_0 = 1-\frac{1}{q}+\epsilon$. A (p,q,s,ϵ) molecule center at x_0 is a function M such that $M(x) \in L^q(\mathbb{R}^n)$ and $|x|^{nb_0}M(x) \in L^q(\mathbb{R}^n)$ satisfying:*

$$(1)\ \|M\|_{L^q}^{a_0} \|M(\cdot)| \cdot -x_0|^{nb_0}\|_{L^q}^{b_0-a_0} < \infty; \quad (2)\int_{\mathbb{R}^n} M(x)x^\alpha dx = 0, 0 \leq |\alpha| \leq s.$$

To prove Theorem 1.13, it suffices to show the followings.

Proposition 3.1. *Let a_Q be a $(p,2,2t)$ atom with $0 < p < 1$ and t be an even integer $t > \frac{n}{p}$.*

1. *If T_a^* defined as in (2) satisfies condition (1) in Theorem 1.13. Then, $T_a^* a_Q$ is a $(p,1,[n(\frac{1}{p}-1)],\frac{t}{n}-\frac{1}{2})$ molecule.*
2. *If T_a defined as in (1) satisfies condition (2) in Theorem 1.13. Then, $T_a a_Q$ is a $(p,1,[n(\frac{1}{p}-1)],\frac{t}{n}-\frac{1}{2})$ molecule.*

Lemma 3.1. *Let $0 < p \leq 1$, $t \geq [n(\frac{1}{p}-1)]$ and a_Q is a $(p,2,2t)$-atom with the center at the origin and $Q = Q(0,l)$ is a cube on which a_Q is supported. Suppose* $0 < l < 1$, $0 \leq \varrho \leq 1$

and $0 \le \delta < 1$. For any positive integer j with $2^j \le l^{-1}$ and any positive integer $2N_1 > \frac{n}{2}$, if $a \in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}$ then

$$\int_{\mathbb{R}^n} |T_j^* a_Q(x)|^q dx \lesssim 2^{jqn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+\varrho(\frac{1}{p}-\frac{1}{q})+(1-\frac{1}{p})\right)} l^{qn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+(1-\frac{1}{p})\right)}; \tag{53}$$

$$\int_{\mathbb{R}^n} |x|^{qt}|T_j^* a_Q(x)|^q dx \lesssim 2^{jqn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+\varrho(\frac{1}{p}-\frac{1}{q})+(1-\frac{1}{p})\right)+jqt(1-\varrho)} l^{qn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+(1-\frac{1}{p})\right)+qt}. \tag{54}$$

Proof. Firstly, we will prove (53). Denote

$$T = l^{\frac{t}{2N_1}} 2^{j\frac{t}{2N_1}-j\varrho},$$

and break up the integral with respect to the variable x as follows

$$\int_{|x|\le 2T} + \int_{|x|>2T}. \tag{55}$$

Next, we show that both of them are bounded by the right hand in 53. Hölder's inequality and Minkowski's inequality show that the first integral in (55) is bounded by:

$$T^{n(1-\frac{q}{2})}\left(\int_{Q(0,l)} |a_Q(y)| \Big(\int_{\mathbb{R}^n} \Big|\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(y,\xi) d\xi\Big|^2 dx\Big)^{\frac{1}{2}} dy\right)^q.$$

Recall that a_Q is a $(p,2,2t)$-atom and $T = l^{\frac{t}{2N_1}} 2^{j\frac{t}{2N_1}-j\varrho}$. Then, the desired estimate can be provided Parseval's identity.

Next, we estimate the second integral in (55). Integrating by parts gives that for any multi-index α with $|\alpha| = N_1$

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} a_j(y,\xi) a_Q(y) dy d\xi \\
=\ & |x|^{-2N_1} \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} (\triangle_\xi)^{\alpha_2}\big(a_j(y,\xi)\big) y^{2\alpha_1} a_Q(y) dy d\xi.
\end{aligned}$$

For any fixed $\xi \in \mathbb{R}^n$, let P_ξ be the Taylor polynomial in y of degree $t-2|\alpha_1|-1$ of

$$e^{-i,\xi\rangle}(\triangle_\xi)^{\alpha_2}\big(a_j(y,\xi)\big)$$

about the origin. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} (\triangle_\xi)^{\alpha_2}\big(a_j(y,\xi)\big) y^{2\alpha_1} a_Q(y) dy \\
=\ & \int_{\mathbb{R}^n} \big(e^{-i\langle y,\xi\rangle}(\triangle_\xi)^{\alpha_2} a_j(y,\xi) - P(y)\big) y^{2\alpha_1} a_Q(y) dy \\
=\ & \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^n} e^{-i\langle \bar{y},\xi\rangle} \xi^{\beta_1} \big(\partial_y^{\beta_2}(\triangle_\xi)^{\alpha_2}\big)\big(a_j(\bar{y},\xi)\big) y^{2\alpha_1+\beta} a_Q(y) dy,
\end{aligned}$$

where $|\beta| = t - 2|\alpha_1|$ and $\bar{y}$ is a point around the origin. Therefore, we can write

$$\begin{aligned}
T_j^* a_Q(x) &= \frac{1}{|x|^{2N_1}} \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} \sum_{|\beta_1|+|\beta_2|=|\beta|} \\
&\quad \times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-\bar{y},\xi\rangle} \xi^{\beta_1} \big(\partial_y^{\beta_2}(\triangle_\xi)^{\alpha_2}\big)\big(a_j(\bar{y},\xi)\big) y^{2\alpha_1+\beta} a_Q(y) d\xi dy.
\end{aligned} \tag{56}$$

By Hölder's inequality and Minkowski's inequality, the second integral in (55) is bounded by:

$$\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}\sum_{|\beta_1|+|\beta_2|=|\beta|}\left(\int_{|x|>T}\frac{1}{|x|^{2N_1(\frac{2q}{2-q})}}dx\right)^{1-\frac{q}{2}}$$
$$\times\quad\left(\int_{Q(0,l)}|a_Q(y)|\Big(\int_{\mathbb{R}^n}|\int_{\mathbb{R}^n}e^{i\langle x-\bar{y},\xi\rangle}\xi^{\beta_1}\big(\partial_y^{\beta_2}(\triangle_\xi)^{\alpha_2}\big)\big(a_j(\bar{y},\xi)\big)y^{2\alpha_1+\beta}d\xi|^2dx\Big)^{\frac{1}{2}}dy\right)^q.$$

Recall that a_Q is a $(p,2,2t)$-atom and $T=l^{\frac{t}{2N_1}}2^{j\frac{t}{2N_1}-j\varrho}$. Then, the desired estimate can be obtained Parseval's identity.

The main idea to prove (54) is writing $|x|^t|T_j^*a_Q(x)|$ as sum of T^* first, then following the same method as above to estimate these operator. To this end, we fixed $|\alpha|=\frac{t}{2}$ in (56) and give a clear relationship between y and $\bar{y}$. We can write

$$\begin{aligned}&e^{-i\langle y,\xi\rangle}(\triangle_\xi)^{\alpha_2}\big(a_j(y,\xi)\big)\\ =\;&P_\xi(y)+C_t\int_0^1(1-\theta)^{t-2|\alpha_2|}\Big(\partial_y^{2\alpha-2\alpha_2}e^{-i\langle\cdot,\xi\rangle}(\triangle_\xi)^{\alpha_2}\big(a_j(\cdot,\xi)\big)\Big)(\theta y)y^{2\alpha-2\alpha_2}d\theta.\end{aligned}$$

The cancellation condition of a_Q gives

$$\begin{aligned}|x|^tT_j^*a_Q(x)\;&=\;C_t\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}\int_0^1(1-\theta)^{t-2|\alpha_2|}\sum_{|\beta_1|+|\beta_2|=|\beta|}\\&\quad\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i\langle x-\theta y,\xi\rangle}\xi^{\beta_1}\big(\partial_y^{\beta_2}(\triangle_\xi)^{\alpha_2}\big)\big(a_j(\theta y,\xi)\big)y^{2\alpha_1+\beta}a_Q(y)d\xi dyd\theta,\end{aligned}$$

where $|\beta|=t-2|\alpha_1|$. Denote $a_{j,\beta_1,\beta_2}(y,\xi)=\xi^{\beta_1}\big(\partial_y^{\beta_2}(\triangle_\xi)^{\alpha_2}\big)\big(a_j(y,\xi)\big)$ and $f_{\alpha_1,\beta,Q}(y)=y^{2\alpha_1+\beta}a_Q(y)$. Then

$$|x|^tT_j^*a_Q(x)\;=\;C_t\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}\int_0^1(1-\theta)^{t-2|\alpha_2|}\theta^{-n}\sum_{|\beta_1|+|\beta_2|=|\beta|}\big(T^*_{a_{j,\beta_1,\beta_2}}f_{\alpha_1,\beta,Q}(\theta\cdot)\big)(x)d\theta,$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n}|x|^{qt}|T_j^*a_Q(x)|^qdx\;&\lesssim\;\Bigg(\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}\int_0^1(1-\theta)^{t-2|\alpha_2|}\theta^{-n}\\&\quad\times\sum_{|\beta_1|+|\beta_2|=|\beta|}\Big(\int_{\mathbb{R}^n}|\big(T^*_{a_{j,\beta_1,\beta_2}}f_{\alpha_1,\beta,Q}(\theta\cdot)\big)(x)|^qdx\Big)^{\frac{1}{q}}d\theta\Bigg)^q,\end{aligned}\tag{57}$$

Notice that $a_{j,\beta_1,\beta_2}\in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+t(1-\varrho)}$ with its normal independent of j,β_1,β_2 and $f_{\alpha_1,\beta,Q}(x)$ satisfies $suppf_{\alpha_1,\beta,Q}\subset Q$, $\int_{\mathbb{R}^n}|f_{\alpha_1,\beta,Q}(x)|dx\le|Q|^{1-\frac{1}{p}+\frac{t}{n}}$ and $\int_{\mathbb{R}^n}f_{\alpha_1,\beta,Q}(y)y^\alpha dy=0, 0\le|\alpha|\le t$, since a_Q is a $(p,2,2t)$ atom.

By the same argument as (53), we can get

$$\begin{aligned}\int_{\mathbb{R}^n}|\big(T^*_{a_{j,\beta_1,\beta_2}}f_{\alpha_1,\beta,Q}(\theta\cdot)\big)(x)|^qdx\;&\lesssim\;\theta^{qn}2^{jqn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+\varrho(\frac{1}{p}-\frac{1}{q})+(1-\frac{1}{p})\right)+jqt(1-\varrho)}\\&\quad\times\;l^{qn\left(\frac{t}{2N_1}(\frac{1}{q}-\frac{1}{2})+(1-\frac{1}{p})\right)+qt}.\end{aligned}$$

By substituting this into (57), the desired estimate can be provided immediately, since that $\int_0^1(1-\theta)^{t-2|\alpha_2|}d\theta\lesssim1$ is always true for $2|\alpha_1|\le t$. □

Lemma 3.2. *Let $Q(0,l)$ be a fixed cube with side length $l<1$. Suppose $0<q<2$, $0<p<1$, $0<\varrho<1$ and $0\le\delta<1$. For any positive integer j with $l^{-1}\le 2^j<l^{-\frac{1}{\varrho}}$ if $a\in S^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}_{\varrho,\delta}$ then*

$$\int_{\mathbb{R}^n}|T_j^*a_Q(x)|^q dx \lesssim 2^{jqn\left(\varrho(\frac{1}{p}-\frac{1}{q})+(1-\frac{1}{p})\right)}l^{qn(1-\frac{1}{p})}; \tag{58}$$

$$\int_{\mathbb{R}^n}|x|^{qt}|T_j^*a_Q(x)|^q dx \lesssim 2^{jqn\left(\varrho(\frac{1}{p}-\frac{1}{q})+(1-\frac{1}{p})\right)-jqt\varrho}l^{qn(1-\frac{1}{p})}. \tag{59}$$

Proof. Firstly, we will prove (58). Break up the integral with respect to the variable x as follows

$$\int_{|x|\le 2^{-j\varrho+1}}+\int_{|x|>2^{-j\varrho+1}}. \tag{60}$$

Hölder's inequality and Parseval's identity show that the first integral in (60) is bounded by:

$$\begin{aligned}
&\left(\int_{|x|\le 2^{-j\varrho}}dx\right)^{1-\frac{q}{2}}\left(\int_{\mathbb{R}^n}|\int_{Q(0,l)}\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)a_Q(y)d\xi dy|^2dx\right)^{\frac{q}{2}} \qquad (61)\\
\le\ & 2^{-jn\varrho(1-\frac{q}{2})}\left(\int_{Q(0,l)}|a_Q(y)|\Big(\int_{\mathbb{R}^n}|\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)d\xi|^2dx\Big)^{\frac{1}{2}}dy\right)^q\\
\lesssim\ & 2^{-jn\varrho(1-\frac{q}{2})}\left(\int_{Q(0,l)}|a_Q(y)|\Big(\int_{\mathbb{R}^n}|a_j(y,\xi)|^2d\xi\Big)^{\frac{1}{2}}dy\right)^q\\
\lesssim\ & 2^{-jn\varrho(1-\frac{q}{2})+jq(-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+\frac{n}{2})}l^{q(n-\frac{n}{p})}.
\end{aligned}$$

By Hölder's inequality, integrating by parts, Parseval's identity and the fact $|x|\sim|x-y|$ follows from $l<1$, $y\in Q(0,l)$ and $|x|>2^{-j\varrho+1}$. The second integral in (60) is bounded by:

$$\begin{aligned}
&\Big(\int_{|x|>2^{-j\varrho+1}}\frac{1}{|x|^{N(\frac{2q}{2-q})}}dx\Big)^{1-\frac{q}{2}}\\
\times\ &\Big(\int_{|x|>2^{-j\varrho+1}}|x|^{2N}|\int_{Q(0,l)}\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)a_Q(y)d\xi dy|^2dx\Big)^{\frac{q}{2}} \qquad (62)\\
\lesssim\ & 2^{-j\varrho q\left(n(\frac{1}{q}-\frac{1}{2})-N\right)}\left(\int_{Q(0,l)}|a_Q(y)|\Big(\int_{\mathbb{R}^n}|x-y|^{2N}|\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)d\xi|^2dx\Big)^{\frac{1}{2}}dy\right)^q\\
\lesssim\ & 2^{-j\varrho q\left(n(\frac{1}{q}-\frac{1}{2})-N\right)}\left(\int_{Q(0,l)}|a_Q(y)|\Big(\int_{\mathbb{R}^n}|\partial_\xi^\alpha a_j(y,\xi)|^2d\xi\Big)^{\frac{1}{2}}dy\right)^q\\
\lesssim\ & 2^{-jq\varrho\left(n(\frac{1}{q}-\frac{1}{2})-N\right)+jq(-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+\frac{n}{2}-\varrho N)}l^{q(n-\frac{n}{p})}.
\end{aligned}$$

The proof of (59) is a little different from (58), that is, the first term in (61) and (62) is

$$\Big(\int_{|x|\le 2^{-j\varrho}}|x|^{t\frac{2q}{2-q}}dx\Big)^{1-\frac{q}{2}} \quad\text{and}\quad \Big(\int_{|x|>2^{-j\varrho+1}}\frac{1}{|x|^{(N-t)(\frac{2q}{2-q})}}dx\Big)^{1-\frac{q}{2}}.$$

□

In the course of the above proof, if $\varrho=0$, $|x|\sim|x-y|$ is still true for $l<1$, $y\in Q(0,l)$ and $|x|>2$. Thus, we have the followings.

Lemma 3.3. *Let $Q(0,l)$ be a fixed cube with side length $l<1$. Suppose $0<q<2$, $0<p<1$ and $0\le\delta<1$. For any positive integer j with $l^{-1}\le 2^j$ if $a\in S_{0,\delta}^{-n(\frac{1}{p}-\frac{1}{2})}$ then*

$$\int_{\mathbb{R}^n}|T_j^*a_Q(x)|^q dx \lesssim 2^{jqn(1-\frac{1}{p})}l^{qn(1-\frac{1}{p})}; \tag{63}$$

$$\int_{\mathbb{R}^n}|x|^{qt}|T_j^*a_Q(x)|^q dx \lesssim 2^{jqn(1-\frac{1}{p})}l^{qn(1-\frac{1}{p})}. \tag{64}$$

Lemma 3.4. *Suppose $0<q<2$, $0<p\le 1$, $0\le\varrho<1$ and $0\le\delta<1$. For any positive integer $2N_2>\frac{n(2-p)}{2}$, if $a\in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})}$ then*

$$\begin{aligned}\int_{\mathbb{R}^n}|T_j^*a_Q(x)|^q dx &\lesssim l^{qn(\frac{1}{q}-\frac{1}{p})}2^{-jq\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)}\\ &+ l^{q\left(n(\frac{1}{q}-\frac{1}{p}+\frac{1}{2})-N_2\right)}2^{jq(-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+\frac{n}{2}-\varrho N_2)},\end{aligned} \tag{65}$$

$$\begin{aligned}\int_{\mathbb{R}^n}|x|^{qt}|T_j^*a_Q(x)|^q dx &\lesssim l^{qn(\frac{1}{q}-\frac{1}{p})+qt}2^{-jq\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)}\\ &+ l^{q\left(n(\frac{1}{q}-\frac{1}{p}+\frac{1}{2})-N_2\right)}2^{jq\left(-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+\frac{n}{2}-\varrho(N_2+t)\right)}.\end{aligned} \tag{66}$$

Proof. Break up the integral with respect to the variable x as follows

$$\int_{|x|\le 2l}+\int_{|x|>2l}. \tag{67}$$

Notice that $a_j(y,\xi)\in S_{\varrho,\delta}^{-\frac{n}{2}\max(0,\delta-\varrho)}$ with bounds $\lesssim 2^{-jn(1-\varrho)(\frac{1}{p}-\frac{1}{2})+j\frac{n}{2}\max(0,\delta-\varrho)}$. Hölder's inequality and the L^2-estimate of T_j give that the first integral in (67) is bounded by:

$$\begin{aligned}&\Big(\int_{|x|\le 2l}dx\Big)^{1-\frac{q}{2}}\|T_j^*a_Q\|_{L^2}^q \\ \le\ & l^{n(1-\frac{q}{2})}2^{-jq\left(n(1-\varrho)(\frac{1}{q}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)}\|a_Q\|_{L^2}^q\\ \lesssim\ & l^{qn(\frac{1}{q}-\frac{1}{p})}2^{-jq\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)}.\end{aligned} \tag{68}$$

By Hölder's inequality, integrating by parts, the Parseval's identity and the fact that $|x|\sim|x-y|$ follows from $l<1$, $y\in Q(0,l)$ and $|x|>2l$. The second integral in (67) is bounded by:

$$\begin{aligned}&\Big(\int_{|x|>2l}\frac{1}{|x|^{N_2(\frac{2q}{2-q})}}dx\Big)^{1-\frac{q}{2}}\\ \times\ &\Big(\int_{|x|>2l}|x|^{2N_2}\Big|\int_{Q(0,l)}\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)a_Q(y)d\xi dy\Big|^2dx\Big)^{\frac{q}{2}}\\ \lesssim\ & l^{q\left(n(\frac{1}{q}-\frac{1}{p}+\frac{1}{2})-N_2\right)}2^{jq(-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})+\frac{n}{2}-\varrho N_2)}.\end{aligned} \tag{69}$$

The proof of (66) is a little different from (65), that is, (68) and (69) in this case are given as:

$$\Big(\int_{|x|\le 2l}|x|^{t\frac{2p}{2-p}}dx\Big)^{1-\frac{p}{2}}\|T_j^*a_Q\|_{L^2}^p$$

and

$$\begin{aligned}&\Big(\int_{|x|>2l}\frac{1}{|x|^{N_2(\frac{2p}{2-p})}}dx\Big)^{1-\frac{p}{2}}\\ \times\ &\Big(\int_{|x|>2l}|x|^{2(N_2+t)}\Big|\int_{Q(0,l)}\int_{\mathbb{R}^n}e^{i\langle x-y,\xi\rangle}a_j(y,\xi)a_Q(y)d\xi dy\Big|^2dx\Big)^{\frac{p}{2}},\end{aligned}$$

respectively. □

Remark 3.1. *Lemma 3.2 and Lemma 3.4 are still valid when $\varrho = 1$, but both of them can not be used. In fact, there is no T_j^* in Lemma 3.2 and no convergence factor in Lemma 3.4. However, the H^p-continuity in this paper can be proved without them.*

Remark 3.2. *Lemma 3.1, Lemma 3.2 and Lemma 3.4 hold for T_j. They can be proved parallelly, provided in the argument above if we apply the L^2-estimate for pseudo-differential operators instead of the Parseval's identity at cost of $a \in S_{\varrho,\delta}^{-n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)}$.*

Proof of Proposition 3.1. (1) is considered firstly. By standard molecular technique, it is sufficient to show that if a_Q be a $(p,2,2t)$ atom with t an even integer $t > \frac{n}{p}$, then $T^* a_Q$ is a $(p,1,s,\epsilon)$ molecule, where $s = [n(\frac{1}{p}-1)]$. Without loss of generality, we assume $Q = Q(0,l)$. Take $\epsilon = \frac{t}{n} - \frac{1}{2}$ (clearly,$\epsilon > \max\{\frac{s}{n}, \frac{1}{p}-1\}$), then $a_0 = 1 - \frac{1}{p} + \frac{t}{n}$ and $b_0 = \frac{t}{n}$. The vanishing of $T^* a_Q$ is clear. So, it has to be shown that

$$\|T^* a_Q\|_{L^1}^{1-\frac{1}{p}+\frac{t}{n}} \||\cdot|^t T^* a_Q(\cdot)\|_{L^1}^{\frac{1}{p}-1} < \infty.$$

To this end, it suffices to show the following inequalities

$$\begin{cases} \|T^* a_Q\|_{L^1} \lesssim l^{\varrho(n-\frac{n}{p})} \quad \text{and} \quad \||\cdot|^t T^* a_Q(\cdot)\|_{L^1} \lesssim l^{\varrho(t+n-\frac{n}{p})}, & \text{if } 0 < l < 1; \\ \|T^* a_Q\|_{L^1} \lesssim l^{(n-\frac{n}{p})} \quad \text{and} \quad \||\cdot|^t T^* a_Q(\cdot)\|_{L^1} \lesssim l^{(t+n-\frac{n}{p})}, & \text{if } l \geq 1; \end{cases}$$

We compose the operator T_a as (10) when $0 \leq \varrho < 1$, then $\|T^* a_Q\|_{L^1}$ and $\||\cdot|^t T^* a_Q(\cdot)\|_{L^1}$ are bounded by:

$$\sum_j \int_{\mathbb{R}^n} |T_j^* a_Q(x)| dx \quad \text{and} \quad \sum_j \int_{\mathbb{R}^n} |x|^t |T_j^* a_Q(x)| dx.$$

Case 1. For $0 < l < 1$; break up this sum as before, that is,

$$\begin{cases} \sum\limits_{2^j < l^{-1}} + \sum\limits_{l^{-1} < 2^j}, & \text{if } \varrho = 0; \\ \sum\limits_{2^j < l^{-1}} + \sum\limits_{l^{-1} \leq 2^j \leq l^{-\frac{1}{\varrho}}} + \sum\limits_{l^{-\frac{1}{\varrho}} < 2^j}, & \text{if } 0 < \varrho < 1. \end{cases} \tag{70}$$

If $0 < \varrho < 1$, by Lemma 3.1, Lemma 3.2 and Lemma 3.4 after taking $q = 1$, the corresponding sum can be bounded by

$$\sum_{2^j < l^{-1}} 2^{jn\varrho(\frac{1}{p}-1)+jn(1-\frac{1}{p})+jn\frac{t}{2N}(1-\frac{p}{2})} l^{n(1-\frac{1}{p})+n\frac{t}{2N}(1-\frac{p}{2})} + \sum_{l^{-1} \leq 2^j \leq l^{-\frac{1}{\varrho}}} 2^{-jn(1-\varrho)(\frac{1}{p}-1)} l^{n(1-\frac{1}{p})}$$

$$+ \sum_{l^{-\frac{1}{\varrho}} < 2^j} \Big(l^{n(1-\frac{1}{p})} 2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)} + l^{n(1-\frac{1}{p})+(\frac{n}{2}-N_2)} 2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}+\varrho N_2\right)} \Big).$$

Clearly, the second sum above is convergent to $l^{\varrho(n-\frac{n}{p})}$ since $n(1-\varrho)(\frac{1}{p}-1) > 0$. Note that t is large enough, and so we can choose suitable positive integer $2N > \frac{n}{2}$ so that $n\varrho(\frac{1}{p}-1) + n(1-\frac{1}{p}) + n\frac{t}{2N}(1-\frac{p}{2}) > 0$ since $1 - \frac{p}{2} > 0$. The first sum is convergent to $l^{\varrho(n-\frac{n}{p})}$ as well. Notice that $n(1-\varrho)(\frac{1}{p}-\frac{1}{2}) - \frac{n}{2}\max(0,\delta-\varrho) > 0$, the first term in last sum is convergent to

$$l^{n(1-\frac{1}{p})+\frac{1}{\varrho}(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho))} = l^{\varrho(n-\frac{n}{p})+\frac{n}{\varrho}\left((\frac{1}{p}-\frac{1}{2})(1-\varrho)^2+\frac{1}{2}(-\varrho^2+\varrho)-\max(0,\delta-\varrho)\right)}.$$

Notice that $l<1$ and

$$\begin{aligned} & \frac{n}{\varrho}\big((\frac{1}{p}-\frac{1}{2})(1-\varrho)^2+\frac{1}{2}(-\varrho^2+\varrho)-\frac{1}{2}\max(0,\delta-\varrho)\big) \\ \geq\ & \frac{n}{\varrho}\big((\frac{1}{p}-\frac{1}{2})(1-\varrho)^2+\frac{1}{2}(-\varrho^2+2\varrho-1)\big) \\ =\ & \frac{n}{\varrho}(\frac{1}{p}-1)(1-\varrho)^2>0. \end{aligned}$$

We can get the first term in last sum is less then $l^{\varrho(n-\frac{n}{p})}$. Taking N_2 large enough, we get the second term in last sum is convergent to

$$l^{n(1-\frac{1}{p})+\frac{n}{2}+\frac{1}{\varrho}(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2})}=l^{\varrho(n-\frac{n}{p})+\frac{n}{\varrho}\left((\frac{1}{p}-1)(1-\varrho)^2\right)}\leq l^{\varrho(n-\frac{n}{p})}.$$

If $\varrho=0$. Lemma 3.1, Lemma 3.2 after taking $q=1$, and Lemma 3.3 give that the corresponding sum can be bounded by:

$$\sum_{2^j<l^{-1}}2^{jn(1-\frac{1}{p})+jn\frac{t}{2N}(1-\frac{p}{2})}l^{n(1-\frac{1}{p})+n\frac{t}{2N}(1-\frac{p}{2})}+\sum_{l^{-1}\leq 2^j}2^{-jn(\frac{1}{p}-1)}l^{n(1-\frac{1}{p})}\lesssim 1$$

If $\varrho=1$, we can divide $a(x,\xi)$, with respect to variate ξ, smoothly into two parts, that is, $a(x,\xi)=\tilde{a}_1(x,\xi)+\tilde{a}_2(x,\xi)$ with $supp_\xi\tilde{a}_1(x,\xi)\subset\{|\xi|\leq l^{-1}\}$ and $supp_\xi\tilde{a}_2(x,\xi)\subset\{|\xi|\geq l^{-1}\}$. For $T^*_{\tilde{a}_1}$, we compose it into $\sum_j T^*_{\tilde{a}_1,j}$ as (10), then $T^*_{\tilde{a}_1,j}a_Q=0$ when $2^j\geq l^{-1}$ and $T^*_{\tilde{a}_1,j}a_Q$ meat the condition of Lemma 3.1 when $2^j\leq l^{-1}$. So

$$\begin{aligned} \int_{\mathbb{R}^n}|T^*_{\tilde{a}_1,j}a_Q(x)|dx &\leq \sum_{2^j\leq l^{-1}}\int_{\mathbb{R}^n}|T^*_{\tilde{a}_1,j}a_Q(x)|dx \\ &\lesssim \sum_{2^j<l^{-1}}2^{jn(\frac{1}{p}-1)+jn(1-\frac{1}{p})+jn\frac{t}{2N}(1-\frac{p}{2})}l^{n(1-\frac{1}{p})+n\frac{t}{2N}(1-\frac{p}{2})}\lesssim l^{(n-\frac{n}{p})}. \end{aligned}$$

Notice that $\tilde{a}_2(y,\xi)\in S^0_{1,\delta}$. By the same argument as Lemma 3.4, it is easy to get

$$\int_{\mathbb{R}^n}|T^*_{\tilde{a}_2}a_Q(x)|dx\lesssim l^{n(1-\frac{1}{p})}.$$

Next, we show the inequality $\||\cdot|^tT^*a_Q(\cdot)\|_{L^1}\lesssim l^{\varrho(t+n-\frac{n}{p})}$. Notice that t is fixed large enough and $l^t\leq l^{\varrho t}$ for $0\leq\varrho\leq 1$, this inequality can be obtained by a similar argument as above. Here, the estimates (54),(59) and (66) will be applied instead of (53), (58) and (65).

Case 2. $l\geq 1$; (65) in Lemma 3.4 gives (after taking $q=1$ and N_2 large enough)

$$\begin{aligned} \sum_j\int_{\mathbb{R}^n}|T^*_ja_Q(x)|dx &\lesssim \sum_j\Big(l^{n(1-\frac{1}{p})}2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)} \\ &+ l^{n(1-\frac{1}{p})+(\frac{n}{2}-N_2)}2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}+\varrho N_2\right)}\Big)\lesssim l^{n(1-\frac{1}{p})}. \end{aligned}$$

(66) in Lemma 3.4 gives (after taking $q=1$ and N_2 large enough)

$$\begin{aligned} \sum_j\int_{\mathbb{R}^n}|x|^t|T^*_ja_Q(x)|dx &\lesssim \sum_j\Big(l^{n(1-\frac{1}{p})+t}2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)} \\ &+ l^{n(1-\frac{1}{p})+(\frac{n}{2}-N_2)}2^{-j\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}+\varrho(N_2+t)\right)}\Big) \\ &\lesssim l^{n(1-\frac{1}{p})+t}+l^{n(1-\frac{1}{p})}\leq l^{n(1-\frac{1}{p})+t}. \end{aligned}$$

By Remark 3.2, the proofs of (2) are completely parallel. $\square$

Proof of Theorem 1.16. The proofs of (1) will be shown only and the proofs of (2) are completely parallel. Here, we always assume $0 \leq \varrho < 1$ as the case $\varrho = 1$ is considered in Theorem 1.15.

The $0 < p < 1$ is considered firstly. Let nonnegative integer $t \geq [n(\frac{1}{p} - 1)]$ ($[x]$ indicates the integer part of $[x]$). A function $a_Q \in L(\mathbb{R}^n)$ is called $(p, 2, t)$ atom if it satisfies the following conditions:

$$(1)\ supp a_Q \subset Q; \quad (2) \int_{\mathbb{R}^n} |a_Q(y)| \leq |Q|^{1-\frac{1}{p}}; \quad (3) \int_{\mathbb{R}^n} a_Q(y) y^\alpha dy = 0, 0 \leq |\alpha| \leq t,$$

where $Q = Q(\bar{y}, l)$ is the cube about $\bar{y}$ with sidelength $l > 0$. According to the characterization of the Hardy spaces $H^p(\mathbb{R}^n)$ via the atomic decomposition, it suffices to show that

$$\int_{\mathbb{R}^n} |T^* a_Q(x)|^p dx \leq C, \tag{71}$$

for an individual $(p, 2, t)$ atom a_Q, where constant C independent of a_Q. We assume without loss of generality the center of the cube Q is at the origin and decompose the operator T_a^* as (10). Then, we have

$$\int_{\mathbb{R}^n} |T^* a_Q(x)|^p dx \quad \leq \quad \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} |T_j^* a_Q(x)|^p dx. \tag{72}$$

For the case $l \geq 1$, Lemma 3.4 (after taking $q = p$) implies that it can be bounded by:

$$\sum_{j=0}^{\infty} \left(2^{-jp\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)} + l^{p(\frac{n}{2}-N_2)} 2^{-jp(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}+\varrho N_2)} \right).$$

Clearly, $n(1-\varrho)(\frac{1}{p} - \frac{1}{2}) - \frac{n}{2}\max(0, \delta - \varrho) > 0$ and $n(1-\varrho)(\frac{1}{p} - \frac{1}{2}) - \frac{n}{2} + \varrho N_2 > 0$ (the case $\varrho = 0$ is trivial and the case $\varrho \neq 0$ can be provided by letting N_2 large enough). So, the sum above is convergence.

Next, we consider on the case $l < 1$. Break up the sum in (72) as (70) again.

If $0 < \varrho < 1$, by Lemma 3.1, Lemma 3.2 and Lemma 3.4 (after taking $q = p$), we see that it can be bounded by:

$$\sum_{2^j < l^{-1}} 2^{jn(p-1)+jn\frac{t}{2N_1}(1-\frac{p}{2})} l^{n(p-1)+n\frac{t}{2N_1}(1-\frac{p}{2})} + \sum_{l^{-1} \leq 2^j \leq l^{-\frac{1}{\varrho}}} 2^{jn(p-1)} l^{n(p-1)}$$
$$+ \sum_{l^{-\frac{1}{\varrho}} < 2^j} \left(2^{-jp\left(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}\max(0,\delta-\varrho)\right)} + l^{p(\frac{n}{2}-N_2)} 2^{-jp(n(1-\varrho)(\frac{1}{p}-\frac{1}{2})-\frac{n}{2}+\varrho N_2)} \right).$$

It is easy to see that the second term above is convergent since $0 < p < 1$. Let t large enough, and then we can choose suitable positive integer $2N_1 > \frac{n(2-p)}{2}$ so that $n(p-1) + n\frac{t}{2N_1}(1 - \frac{p}{2}) > 0$ since $1 - \frac{p}{2} > 0$. The first term is convergent too. Taking N_2 large enough, we get the last term is convergent to

$$1 + l^{n(1-p)(\frac{1}{\varrho}-1)} \lesssim 1.$$

If $\varrho = 0$, by Lemma 3.1, Remark 3.3 and the same argument as above, we get the desired estimate easily.

Now, we consider that $p = 1$. The case $0 \leq \delta \leq \varrho < 1$ has been performed in [33]. The remaining case $0 \leq \varrho < \delta < 1$ will be considered only. To this end, break the sum in (72) as (70) again. The sum for $2^j < l^{-1}$ and $2^j > l^{-\frac{1}{1-\delta}}$ when $\varrho = 0$, and for $2^j < l^{-1}$ and $2^j > l^{-\frac{1}{\varrho}}$ when $0 < \varrho < 1$ are convergence by Lemma 3.1 and Lemma 3.4 (after taking $q = p = 1$). By Lemma 3.2, one can not deal with the sum for $l^{-1} \leq 2^j \leq l^{-\frac{1}{1-\delta}}$ when $\varrho = 0$, and for $l^{-1} \leq 2^j \leq l^{-\frac{1}{\varrho}}$ when $0 < \varrho < 1$ as above. Because, there is no convergence factor in this lemma when $q = p = 1$. One can overcome this problem as the corresponding case in the proof of Theorem 1.5 by following lemmas. The proof is completed. □

Lemma 3.5. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $0 < \varrho < \delta < 1$, $a \in S_{\varrho,\delta}^{-\frac{n}{2}(1-\varrho)}$. Then for any $1 \le \lambda \le \frac{1}{\varrho}$, any positive integer $N > \frac{n}{2}$ and any positive integer j with $l^{-\lambda} \le 2^j \le l^{-\frac{1}{\varrho}}$, we have*

$$\int_{\mathbb{R}^n} |T_j^* a_Q(x)| dx \lesssim 2^{j\delta} l^{\lambda} + 2^{j\frac{n}{2}(\frac{n}{2N}-1)} l^{\frac{n\lambda}{2}(\frac{n}{2N}-1)}.$$

Lemma 3.6. *Let $Q(x_0, l)$ be a fixed cube with side length $l < 1$. Suppose $\varrho = 0$, $0 < \delta < 1$, $a \in S_{0,\delta}^{-\frac{n}{2}}$, then for any $1 \le \lambda \le \frac{1}{1-\delta}$, any positive integer $N > \frac{n}{2}$ and any positive integer j with $l^{-\lambda} \le 2^j \le l^{-\frac{1}{1-\delta}}$,*

$$\int_{\mathbb{R}^n} |T_j^* a_Q(x)| dx \lesssim 2^{j\delta} l^{\lambda} + 2^{j\frac{n}{2}(\frac{n}{2N}-1)} l^{\frac{n\lambda}{2}(\frac{n}{2N}-1)}.$$

These lemmas can be proved by the main idea in the proof Lemma 2.5. We will only outline the proof of Lemma 3.5.

Proof of Lemma 3.5. Let $Q(x_i, l^\lambda)$ be given as in the proof of Lemma 2.5.

$$Q(x_0, l) \subset \cup_{i=1}^{L^n} Q(x_i, l^\lambda) \subset Q(x_0, 2l).$$

Denote

$$T_{j,i}^* a_Q(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x_i, \xi) \psi(2^{-j}\xi) d\xi a_Q(y) dy.$$

We can write

$$\int_{\mathbb{R}^n} |T_j^* a_Q(x)| dx \le \sum_{i=1}^{L^n} \int_{\mathbb{R}^n} |T_j^* (a_Q \chi_{Q(x_i,l^\lambda)})(x)| dx$$
$$\le \sum_{i=1}^{L^n} \Big(\int_{\mathbb{R}^n} |T_j^*(a_Q\chi_{Q(x_i,l^\lambda)})(x) - T_{j,i}^*(a_Q\chi_{Q(x_i,l^\lambda)})(x)| dx + \int_{\mathbb{R}^n} |T_{j,i}^*(a_Q\chi_{Q(x_i,l^\lambda)})(x)| dx \Big).$$

Using the similar method as Lemma 2.3 ($p = 2$) and Lemma 2.4 ($p = 2$), one can get

$$\int_{\mathbb{R}^n} |T_j^*(a_Q\chi_{Q(x_i,l^\lambda)})(x) - T_{j,i}^*(a_Q\chi_{Q(x_i,l^\lambda)})(x)| dx \lesssim 2^{j\delta} l^{n(\lambda-1)+\lambda}$$

and

$$\int_{\mathbb{R}^n} |T_{j,i}^*(a_Q\chi_{Q(x_i,l^\lambda)})(x)| dx \lesssim 2^{j\frac{n}{2}(\frac{n}{2N}-1)} l^{\frac{n\lambda}{2}(\frac{n}{2N}-1)+n(1-\lambda)},$$

which gives the desired estimate immediately. □

4. Conclusion

This paper has investigated the boundedness properties of pseudo-differential operators and their adjoints on various function spaces. We established pointwise estimates involving the sharp maximal function, which in turn yielded weighted L^p and Hardy space H^p ($0 < p \le 1$) inequalities. A key improvement lies in extending the range of parameters ϱ, δ to the general case $0 \le \varrho \le 1$, $0 \le \delta < 1$ and obtaining results for the adjoint operator T_a^*, whose L^p theory differs from that of T_a. For the Hardy space H^p, we introduced a generalized cancellation condition to prove boundedness for a wider range of p than previously known. The results significantly extend classical theorems proved by several authors providing a more complete picture of the mapping properties for these operators in the Hörmander class.

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